J. Fluid Mech. (1976), *vol.* 78, *part* 1, *pp.* 33–48
Printed in Great Britain

The effective thermal conductivity of sheared suspensions

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(Received 19 July 1975 and in revised form 21 June 1976)

Formal expressions are derived for the effective thermal conductivity K_{ij} of randomly dispersed suspensions undergoing shear. These are then evaluated for the cases of dilute suspensions of cylinders and of spheres when the bulk motion is a simple shear, the Péclet number Pe is large, and the particle Reynolds number is small enough for inertia effects to be negligible. It is shown that as $Pe \rightarrow \infty$ the presence of shear can significantly affect the $O(\phi)$ contribution to K_{ij} (ϕ being the volume fraction of the solids), which becomes independent of k^*, the thermal conductivity of the suspended material. This results from the presence of regions of closed streamlines surrounding each particle which, for sufficiently large Pe, attain an isothermal state and therefore act as regions of infinite conductivity.

1. Introduction

Many studies have appeared in the literature over the past few years dealing with the transport properties of suspensions or of composites containing random dispersions of small particles under conditions where the heterogeneous system behaves like a homogeneous material on a macroscale. Their aim has been to derive expressions for the effective or bulk parameters of such composites given the properties of the individual phases as well as the geometry and spatial distribution of the particles in the suspending medium. Familiar examples include studies of the effective viscosity of suspensions of solid spheres, the rheology of emulsions, the effective thermal, magnetic and elastic properties of composite materials, and many others, all of which can generally be developed within a common theoretical framework. This has been illustrated recently by Batchelor (1974), who has also reviewed some of the more significant contributions to the subject.

The problem of determining K_{ij}, the effective thermal conductivity of suspensions, with which we shall be concerned here, appears to have been first posed by Maxwell, who in 1873 successfully treated the case of an infinitely dilute dispersion of solid spheres. Maxwell's result was then extended by numerous investigators, and most recently by Rocha & Acrivos (1973), who gave expressions for K_{ij} correct to $O(\phi)$ for dilute composites containing inclusions of arbitrary shape (ϕ = volume fraction of solids in the suspension). With the exception of a recent study by Leal (1973), however, all previous investigations on this topic

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have been restricted to stationary media with the result that little is known regarding the extent to which K_{ij} can be enhanced by the presence of relative motion between the two phases comprising the system. Here we shall investigate this effect: specifically, the degree to which the $O(\phi)$ contribution to K_{ij} will be affected by the presence of convection in a flowing suspension.

Our analysis will adopt a point of view which, by now, is a conventional one in the field of suspension rheology (Batchelor 1970). Specifically, we suppose that there exists in the sample a volume V which contains a statistically significant number of particles and whose linear dimensions, $O(V^{\frac{1}{3}})$, are much larger than l, the characteristic dimension of the individual particles, but much smaller than the macroscale L over which significant bulk temperature variations take place. Each variable in the domain, e.g. the temperature T, is then expressed as the sum of a bulk quantity $\overline{T}$ and a fluctuation t', where $\overline{T}$ is defined as the average of the local temperature over the sample volume V, or as an ensemble average over a statistically significant number of realizations. An important consequence of the inequality $l \ll V^{\frac{1}{3}} \ll L$ is that the bulk variables thus defined are independent of the size and shape of V and vary over distances $O(L)$, and that the fluctuating quantities vary randomly over distances $O(l)$ owing to the assumed random location of the particles in the dispersion. The bulk transport coefficients can then be obtained, in principle at least, in a straightforward manner after the fluctuating quantities have first been determined from the solution, on a microscale, of the appropriate conservation equations.

Using this approach, we derive in the next section formal expressions for the bulk heat flux Q_i and for the corresponding effective thermal conductivity K_{ij} which include the convective effects in a flowing suspension. Then, in §§3 and 4 we obtain some results for a dilute suspension of cylinders or of spheres when the bulk motion is a simple shear flow and when inertia effects on the scale of the particle size l are negligible. In contrast to Leal (1973), who derived an expression for the transverse conductivity of a sheared dilute suspension of spherical drops in the limit of low Péclet number Pe, we shall deal here only with the case $Pe \gg 1$, where the effects due to convection are expected to be most significant. We also note that the present analysis applies, of course, equally well to problems in mass transfer provided only that the appropriate terminology is employed.

2. General theory

Consider a suspension of neutrally buoyant particles which is homogeneously dispersed in a viscous incompressible fluid in the presence of a bulk shear and a bulk temperature field. We wish to construct a macroscopic heat conservation equation and a constitutive relation for the various bulk quantities which describe the suspension as an equivalent homogeneous material. Hence, in the absence of viscous dissipation terms, the desired energy equation for the suspension is

$$\frac{\partial H}{\partial \tau} + U_i \frac{\partial H}{\partial x_i} + \frac{\partial Q_i}{\partial x_i} = 0, \tag{2.1}$$

with H being the bulk enthalpy, U_i the bulk velocity and Q_i the bulk conductive heat flux. For the case of a time-independent, linearly varying bulk temperature field, with which we shall be exclusively concerned here, Q_i is related to the bulk temperature gradient $\partial \overline{T}/\partial x_j$ via a Fourier law

$$Q_i + K_{ij}\partial \overline{T}/\partial x_j = 0, \tag{2.2}$$

K_{ij} being referred to as the effective thermal conductivity of the suspension. As may be seen by substituting (2.2) into the steady-state form of (2.1), a linearly varying time-independent bulk temperature field can exist only if $U_i \partial H/\partial x_i \equiv 0$, a condition which we shall henceforth assume to be satisfied. Thus it should be carefully kept in mind that K_{ij} is *not* a tensor, in spite of the notation used here, because, for a given orientation of the bulk velocity U_i, (2.2) applies only for a restricted class of bulk temperature fields.

The microscopic heat equation describing the pointwise energy balance in either the continuous fluid or the inclusions is of course

$$\frac{\partial h}{\partial \tau} + u_i \frac{\partial h}{\partial x_i} + \frac{\partial q_i}{\partial x_i} = 0, \tag{2.3}$$

where h, u_i and q_i all denote local quantities. As mentioned in the introduction, the velocity and the enthalpy in (2.3) are now expressed as the sum of the appropriate bulk quantities and the corresponding local fluctuations u_i' and h', and (2.3) is averaged either over V or over many realizations. Then, taking into account that $\langle h' \rangle = \langle u_i' \rangle = 0$, where the brackets denote the averaging operator, we arrive at the bulk conservation relation

$$\frac{\partial H}{\partial \tau} + U_i \frac{\partial H}{\partial x_i} + \frac{\partial}{\partial x_i} \langle q_i + u_i' h' \rangle = 0, \tag{2.4}$$

which, by comparison with (2.1), readily identifies the bulk conductive heat flux as

$$Q_i \equiv \langle q_i \rangle + \langle u_i' h' \rangle. \tag{2.5}$$

It should be noted here that, for systems containing non-spherical inclusions, the product $u_i' h'$ may in general be time dependent since, for example, elongated particles rotating periodically in a simple shear field spend most of each period aligned with the direction of the undisturbed flow. Consequently, in cases where ensemble averages are replaced by volume averages, the sample volume V should contain a statistically significant number of particles having a representative distribution of orientations, while if time averages are employed to compute the bulk quantities, the averaging time should be large enough to contain a statistically significant number of the rotation periods mentioned above.

In terms of the temperature and the physical properties of the individual phases, the convective term in (2.5) becomes

$$\langle u_i' h' \rangle = \frac{\rho c_p}{V} \int_{V-\Sigma V^*} u_i' t' \, dV + \frac{\rho c_p^*}{V} \int_{\Sigma V^*} u_i' t' \, dV^*, \tag{2.6}$$

where V is again the sample volume, V^* denotes the volume of an inclusion within V, and ρ, c_p and c_p^*, assumed temperature independent, are, respectively, the density and the heat capacities of the continuous and dispersed phases. Then, in view of Fourier's law and (2.6), (2.5) reduces to

$$Q_i \equiv -K_{ij}\frac{\partial \overline{T}}{\partial x_j} = -k\frac{\partial \overline{T}}{\partial x_i} + \frac{k-k^*}{V}\int_{\Sigma V^*}\frac{\partial T}{\partial x_i}dV^* + \rho\frac{c_p^*-c_p}{V}\int_{\Sigma V^*}u_i' t' dV + \rho c_p\langle u_i' t'\rangle, \tag{2.7}$$

where k and k^* are the thermal conductivities of the individual phases. Equation (2.7) explicitly defines K_{ij} in the general case. It suggests, at first perhaps rather surprisingly, that even when the physical properties of the continuous and dispersed phases are identical the existence of a shear field may still alter the effective thermal conductivity from that of the pure materials. Indeed, in his recent study of the effective conductivity of a suspension containing spherical droplets in a simple shear field, Leal (1973) has confirmed this prediction for cases in which the particle Péclet number Pe is small, and has shown that, when $k = k^*$, the transverse conductivity, denoted here by K_{22}, is given by

$$\frac{K_{22}}{k} = 1 + 0\cdot 12\left(\frac{2+5\nu}{1+\nu}\right)^2 \phi Pe^{\frac{3}{2}}, \tag{2.8}$$

where $Pe \equiv \rho c_p a^2 G/k$, G being the strength of the bulk rate of strain, a the radius of the droplets and ϕ their volume fraction, and $\nu \equiv \mu^*/\mu$ the ratio of the viscosities. Evidently, since (2.8) applies only when $Pe \ll 1$, the increase in $\kappa_{22} \equiv (K_{22}/k-1)/\phi$ is small under the above conditions, but appreciable effects are to be expected when the Péclet number is large. In fact, it will be shown in the following sections that, in some cases and for sufficiently large Pe, the components of κ_{ij} become independent of the thermal conductivity of the dispersed phase and attain values relative to those for pure heat conduction which are much larger than in the corresponding case of small Pe.

We also wish to note that the bulk enthalpy H is not necessarily related to the bulk temperature by $H = \rho\langle c_p\rangle \overline{T}$, with $\langle c_p\rangle$ being the volume-weighted average heat capacity, since from the definition

$$H = \frac{1}{V}\int_V h\, dV = \frac{1}{V}\int_V \rho c_p T\, dV, \tag{2.9}$$

it follows readily that

$$H = \rho\{c_p(1-\phi) + c_p^*\phi\}\overline{T} + \rho\frac{c_p^*-c_p}{V}\int_{\Sigma V^*} t'\, dV^*. \tag{2.10}$$

Consequently, the macroscopic heat equation cannot generally be expressed in terms of the bulk temperature gradient, i.e. as

$$\rho\langle c_p\rangle\left\{\frac{\partial \overline{T}}{\partial \tau} + u_i\frac{\partial \overline{T}}{\partial x_i}\right\} + \frac{\partial Q_i}{\partial x_i} = 0, \tag{2.11}$$

unless either

$$\int_{\Sigma V^*} t'\, dV^* = 0$$

or, of course, $c_p = c_p^*$. It is of interest to note that, in the examples of §§3 and 4, (2.11) is satisfied in a steady state.

We shall now consider two relatively simple cases for which some definitive results can be obtained. These will be derived by determining the corresponding temperature field around an isolated particle placed in an infinite domain and then applying (2.7). Although this is a standard practice in suspension rheology for dilute systems, to which the present analysis is confined, we wish to note, nevertheless, that such a procedure replaces the statistically random fields by deterministic solutions, the result being that the disturbance variables often do not vanish when averaged over any representative volume. This difficulty arises in other problems as well, for example the evaluation of the first-order interaction effect in the sedimentation rate of spheres in a quiescent fluid (Batchelor 1972) or the calculation of the $O(\phi^2)$ term in the expression for the bulk stress of a suspension of spherical particles (Batchelor & Green 1972), and must be properly taken into account in the theory.

3. Infinite cylinders in a simple shear

The temperature field

Let us consider a simple shear flow past an infinite circular cylinder, freely rotating and with its axis normal to the plane of flow, in the presence of a bulk temperature field linear in x_2, so that $U_i \partial \overline{T}/\partial x_i = 0$. If inertia effects on the scale of the cylinder radius are assumed negligible, the stream function is given by (e.g. Cox, Zia & Mason 1968)

$$\psi = \tfrac{1}{2}\left\{x_2^2 - \frac{2x_2^2}{r^2} - \frac{1}{2r^2} + \frac{x_2^2}{r^4}\right\}, \quad u_i = \epsilon_{ij3}\frac{\partial \psi}{\partial x_j}, \tag{3.1}$$

where u_i is the velocity, x_i is the position vector and r is the radial distance, all relative to an origin at the centre of the cylinder. The velocities and distances have been normalized such that $u_i = \delta_{i1} x_2$ at infinity and $r = 1$ on the cylinder, where the stream function has been set equal to $-\frac{1}{4}$.

As shown by Cox *et al.* (1968), the flow field consists of two regions. In the first, where $-\frac{1}{4} \leqslant \psi < 0$, all streamlines are closed and a fluid element on a streamline orbits along a constant trajectory indefinitely; in the other, for which $0 \leqslant \psi < \infty$, all streamlines are open and originate at infinity. It should be noted for future reference that the region of closed streamlines extends to infinity in both directions along the x_1 axis.

As the bulk temperature has been set equal to zero at the origin, $\overline{T} = x_2$ [this satisfies the condition $U_i \partial H/\partial x_i \equiv 0$, briefly discussed following (2.2)], which serves as the boundary condition at infinity for the temperature field near the cylinder. This temperature field satisfies the following heat equation, expressed in terms of ψ and an orthogonal co-ordinate η with metric coefficient g:

$$\frac{\partial T}{\partial \eta} = \frac{1}{Pe}\left\{\frac{\partial}{\partial \psi}\left(qg\frac{\partial T}{\partial \psi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{qg}\frac{\partial T}{\partial \eta}\right)\right\}, \tag{3.2}$$

q being the speed and Pe the Péclet number $\rho c_p a^2 G/k$, with a the radius of the cylinder.

We next seek a solution to (3.2) for the case of large Pe by means of a regular perturbation expansion in both the closed- and the open-streamline region; i.e., with Pe^{-1} the small parameter, we suppose that

$$T = T_0 + Pe^{-1} T_1 + \dots \quad . \tag{3.3}$$

In view of (3.2), it is clear that T_0 is a function of ψ only which, for $\psi > 0$, can easily be determined from the known temperature profile at infinity. For the region of closed streamlines, however, such 'upstream information' is of course not available and we must resort to a compatibility condition (cf. Pan & Acrivos 1968) which requires that, in a steady state and in the absence of any sources or sinks in the fluid, the integrated heat flux across any closed streamline be a constant. In our case this constant is simply zero and, hence, taking into account the symmetry of the system, we conclude immediately that, to all orders in Pe^{-1}, $T \equiv 0$ within the cylinder and for $-\frac{1}{4} \leqslant \psi < 0$ except, possibly, in a thin layer along $\psi = 0$. In other words, to this order of approximation, the value of T within the cylinder plus the region of closed streamlines equals that of the bulk temperature at the origin.

This result has some rather striking consequences. First of all, it is clear from (2.7) that, since the particle is embedded in a region of constant temperature, the bulk conductive heat flux, and therefore K_{ij}, will be independent of k^*, i.e. perfectly conducting particles will have exactly the same effect on K_{ij} as insulating inclusions. Moreover, since a region of constant temperature effectively corresponds to a region composed of a perfect conductor, it is anticipated that its existence will be reflected in a significant enhancement of the $O(\phi)$ term in the expression for K_{ij} relative to that for an equivalent stationary suspension.

Returning now to the solution of our problem, we see that for $0 < \psi < \infty$, i.e. in the region of open streamlines,

$$T_0 = \pm (2\psi)^{\frac{1}{2}}, \tag{3.4}$$

where the positive and negative signs refer, respectively, to the upper and lower half-planes. Thus, upon substituting T_0 in (3.3) we obtain the equation for T_1:

$$\frac{\partial T_1}{\partial \eta} = \pm \frac{1}{2^{\frac{1}{2}}} \frac{\partial}{\partial \psi} \left(\frac{qg}{\psi^{\frac{1}{2}}} \right) = \pm \frac{1}{2^{\frac{1}{2}}} \frac{\partial}{\partial \psi} \left(\frac{qg - (2\psi)^{\frac{1}{2}}}{\psi^{\frac{1}{2}}} \right),$$

which upon integration becomes, in the upper half-plane,

$$T_1(\psi, \eta) = T_1(\psi, 0) + \frac{1}{2^{\frac{1}{2}}} \frac{\partial}{\partial \psi} \int_0^\eta \{qg - (2\psi)^{\frac{1}{2}}\} d\eta / \psi^{\frac{1}{2}}, \tag{3.5}$$

where $\eta = 0$ denotes the x_2 axis.

We note, for further use, that, as can easily be shown, the integral

$$\int_0^\infty \{qg - (2\psi)^{\frac{1}{2}}\} d\eta = \int_{R(\psi)}^\infty \left\{ q - \frac{(2\psi)^{\frac{1}{2}}}{g} \right\} \frac{q}{u_r} dr, \tag{3.6}$$

in which u_r is the r velocity component and $R(\psi)$ the value of r along the x_2 axis, is absolutely convergent for all $\psi > 0$. It is evident, however, that (3.3) is not a uniformly valid expansion in Pe^{-1} for all $\psi > 0$ since T_1 becomes singular as $\psi \to 0$. The existence of this singularity indicates, of course, the presence of a thin thermal layer along $\psi = 0$, where the convective and the conductive terms of (3.2) are of comparable magnitude owing to the large temperature gradients across the limiting streamline. Within this layer the curvature and the conduction terms of (3.2) along $\psi = 0$ play a secondary role, so that, after stretching the variables in the customary manner, we obtain

$$\frac{\partial T}{\partial \eta} = \frac{\partial}{\partial \Psi}\left(qg\frac{\partial T}{\partial \Psi}\right), \quad \Psi \equiv Pe^{\frac{1}{2}}\psi, \tag{3.7}$$

with boundary conditions

$$T \to Pe^{-\frac{1}{2}}(2\Psi)^{\frac{1}{2}} \quad \text{as} \quad \Psi \to \infty, \qquad T \to 0 \quad \text{as} \quad \Psi \to -\infty. \tag{3.8}$$

Therefore we need to distinguish three temperature regions: (*a*) an isothermal region of closed streamlines where $-\frac{1}{4} \leqslant \psi < 0$; (*b*) an 'outer' region $\epsilon \leqslant \psi \leqslant \infty$, with ϵ small and fixed, where convection predominates and where T is given by (3.2), (3.4) and (3.5); and (*c*) a boundary-layer-type region of lateral dimensions $O(Pe^{-\frac{1}{2}})$ where T satisfies (3.7) and (3.8). As we shall see presently, this boundary layer plays an important role in the determination of the transverse conductivity K_{22}.

The effective thermal conductivity

As indicated in (2.7), K_{ij} is found from the bulk conductive heat flux, which, in this case, reduces to

$$-Q_i = K_{i2}\frac{\partial \overline{T}}{\partial x_2} = k\left\{\delta_{i2} - \frac{Pe}{V}\int_{V-\Sigma V^*} u_i' t' \, dV - \frac{Pe c_p^*}{V c_p}\int_{\Sigma V^*} u_i' t' \, dV^*\right\} \tag{3.9}$$

because the temperature is constant within each particle. Moreover, for dilute suspensions for which particle–particle interactions can be assumed negligible, the integrals in (3.9) become simply

$$nPe\int_{V_1-V^*} u_i' t' \, dV, \quad \frac{nc_p^*}{c_p} Pe\int_{V^*} u_i' t' \, dV^* \tag{3.10}$$

respectively, where n is the particle concentration, $V_1\,(\gg a^3)$ is a volume within V containing a single inclusion, and V^* refers to the space within a single cylinder. In addition, though,

$$t' = -x_2, \quad u_i' = -\tfrac{1}{2}(\delta_{i1}x_2 + \delta_{i2}x_1)$$

within V^*, and hence

$$n\int_{V^*} u_i' t' \, dV^* = \frac{\phi}{8}\,\delta_{i1},$$

where ϕ is the volume fraction of the solids in the suspension. Therefore the expression for the K_{i2}, which we emphasize once again are not the components of a tensor, becomes in a simple shear flow and for $Pe \to \infty$

$$\frac{K_{i2}}{k} = \delta_{i2} - \frac{\phi c_p^*}{8 c_p} Pe\, \delta_{i1} - nPe\int_{V_1-V^*} u_i' t' \, dV. \tag{3.11}$$

We next turn to the evaluation of K_{22}, which is the most significant and interesting component of K_{ij} since its value reflects the enhancement of the heat flux perpendicular to the direction of the flow resulting from the effects of convection. This requires the determination of the last term in (3.11) with

$$u_2' = -\frac{x_1}{r^4}\left\{2x_2^2+\frac{x_1^2}{2r^2}-\frac{3x_2^2}{2r^2}\right\}, \quad r^2 \equiv x_1^2+x_2^2. \tag{3.12}$$

As mentioned earlier in this section, there exist three distinct regions of the temperature field as $Pe \to \infty$. In the region of closed streamlines, $-\frac{1}{2} \leqslant \psi < 0$, we have again that $t' = -x_2$ to all orders in Pe^{-1}, and hence the corresponding part of the integral in (3.11) becomes

$$-\int_{-\pi}^{\pi}\int_{1}^{R(\theta)} u_2' x_2 r\,dr\,d\theta, \tag{3.13}$$

where $r = R(\theta) = R(-\theta) = R(\pi-\theta)$ denotes the limiting streamline $\psi = 0$. We further note that, along $\psi = 0$,

$$x_2^2 = (2r^2)^{-1}\{1+O(r^{-2})\} \quad \text{as} \quad r \to \infty,$$

and that
$$u_2' = (-x_1/r^4)\{1+O(r^{-2})\} \quad \text{as} \quad r \to \infty$$

within the region of closed streamlines. The integral in (3.13) is, therefore, absolutely convergent and its value is easily seen to be zero owing to the symmetry of $R(\theta)$ and the fact that u_2' is odd in x_1 and even in x_2. Thus, for the purposes of evaluating the last term in (3.11), we need only consider the contribution from the 'outer' region $\epsilon \leqslant \psi \leqslant \infty$ plus that of the thin thermal layer along the $\psi = 0$ streamline.

Let us then turn to the outer region, where in view of (3.3)–(3.5) we have, in the upper half-plane $x_2 \geqslant 0$,

$$t' = (2\psi)^{\frac{1}{2}} - x_2 + \frac{1}{Pe}\left\{T_1(\psi, 0) + \frac{1}{2^{\frac{1}{2}}}\frac{\partial}{\partial\psi}\int_0^{\eta}\{qg-(2\psi)^{\frac{1}{2}}\}\,d\eta/\psi^{\frac{1}{2}}\right\}+\dots. \tag{3.14}$$

The first integral to be considered is therefore

$$\int_{V^+} u_2'\{(2\psi)^{\frac{1}{2}}-x_2\}\,dV, \tag{3.15}$$

where V^+ refers to the outer region $\epsilon \leqslant \psi \leqslant \infty$. However, in view of (3.1) and (3.12)

$$u_2'\{(2\psi)^{\frac{1}{2}}-x_2\} \to 2x_1x_2^3/r^6+O(r^{-4}) \quad \text{as} \quad r\to\infty, \tag{3.16}$$

and hence the integral (3.15) is only conditionally convergent as $r \to \infty$, its value being determined by the order of integration as well as by the size and shape of V^+. This difficulty, which, as mentioned in the introduction, is also encountered in many other studies involving two-phase systems (Batchelor 1974), arises from the fact that, in replacing the term

$$\frac{Pe}{V}\int_{V-\Sigma V^*} u_i' t'\,dV \tag{3.17}$$

of (3.9) by the first integral in (3.10), particle–particle interactions have been assumed negligible to a first approximation. Unfortunately, owing to the slowness with which the disturbance variables u_2' and t' approach zero with increasing r, this assumption is not quite valid, so that the presence of the other particles in the suspension, in addition to the reference cylinder, must be considered in evaluating (3.9) even when $n \to 0$. A method for achieving this, recently proposed by Batchelor (1972), is to look for a disturbance quantity which has the same asymptotic dependence relative to the position of one cylinder as (3.16) and whose bulk value is known, and then to modify the integrand in (3.17) such that the integral is absolutely convergent. In the present case, such a disturbance quantity is

$$\tfrac{1}{4}\,\partial u_1'/\partial x_1 - \tfrac{1}{4}p', \tag{3.18}$$

where $p' = -4x_1x_2/r^4$ is the dimensionless pressure obtained from the creeping-flow equation

$$\partial p'/\partial x_i = \nabla^2 u_i'.$$

Therefore, since the integral of (3.18) over V^* vanishes, and since

$$\langle \partial u_1'/\partial x_1 - p' \rangle = 0,†$$

we can replace (3.17), with $i = 2$, by

$$\frac{Pe}{V}\int_{V-\Sigma V^*}\left[u_2'\{(2\psi)^{\frac{1}{2}} - x_2\} - \frac{1}{4}\frac{\partial u_1'}{\partial x_1} + \tfrac{1}{4}p'\right] dV,$$

which is absolutely convergent. Hence, noting that the integral of (3.18) over the region of closed streamlines also vanishes, the above reduces, as $n \to 0$, to

$$2nPe\int_0^{\pi}\int_{R(\theta)}^{\infty}\left[u_2'\{(2\psi)^{\frac{1}{2}} - x_2\} - \frac{1}{4}\frac{\partial u_1'}{\partial x_1} + \tfrac{1}{4}p'\right] r\,dr\,d\theta,$$

which is easily seen to vanish since u_2', $\partial u_1'/\partial x_1$ and p' are odd in x_1 while $(2\psi)^{\frac{1}{2}}$ is even. Of course, this result should have been anticipated because, since the integral in (3.15) should be unaffected by a reversal in the flow direction, its value must be zero. The argument presented above is, however, somewhat more satisfying and leaves no doubt regarding the final conclusion.

Let us next evaluate the leading contribution to K_{22} arising from the thermal layer. We note first of all that, in view of the small thickness of this layer, (3.7) and (3.8) can also be expressed as

$$\frac{\partial T}{\partial \eta} = q_0 g_0 \frac{\partial^2 T}{\partial \Psi^2}, \qquad T \to \frac{1}{Pe^{\frac{1}{2}}}(2\Psi)^{\frac{1}{2}} \quad \text{as} \quad \Psi \to \infty, \tag{3.19}$$

where $q_0(\eta)$ and $g_0(\eta)$ refer to the corresponding variables calculated along the limiting streamline $\psi = 0$. Therefore, with $\mathscr{H}(\Psi)$ denoting the Heaviside function, we have that, within the thermal layer,

$$T - \frac{\mathscr{H}}{Pe^{\frac{1}{2}}}(2\Psi)^{\frac{1}{2}} = \left\{T - \frac{\mathscr{H}}{Pe^{\frac{1}{2}}}(2\Psi)^{\frac{1}{2}}\right\}_{\eta=0} + \int_0^{\eta} q_0 g_0 \frac{\partial^2 T}{\partial \Psi^2}\, d\eta.$$

† Because of symmetry, p', defined everywhere as $-\frac{1}{2}\sigma_{kk}'$, with σ_{ij}' the fluctuating stress, integrates to zero within the particle.

However, since u_2' is also approximately constant and equal to u_{20}' across the thermal layer, we have that as $\Psi\to\infty$

$$Pe^{-\frac{1}{3}}\int_{-\infty}^{\Psi}\frac{u_2'}{q}\left\{T-\frac{\mathscr{H}}{Pe^{\frac{1}{3}}}(2\Psi)^{\frac{1}{2}}\right\}d\Psi\to Pe^{-\frac{2}{3}}\frac{u_{20}'}{q_0}C(\Psi)-\frac{Pe^{-\frac{2}{3}}}{(2\Psi)^{\frac{1}{2}}}\frac{u_{20}'}{q_0}\int_0^{\eta}q_0\,g_0\,dz, \quad (3.20)$$

where for the purpose of evaluating K_{22} we can omit the function $C(\Psi)$ because the absolutely convergent integral

$$\int_{-\infty}^{\infty}\frac{u_{20}'}{q_0}g_0\,d\eta = 0.$$

Hence the first non-zero contribution to the $O(\phi)$ term of K_{22} results from the function T_1 of (3.5) plus the second term of (3.20). Specifically, since the absolutely convergent integral

$$\int_{-\infty}^{\infty}\int_{\epsilon}^{\infty}\frac{u_2'g}{q}T_1(\psi,0)\,d\psi\,d\eta = 0$$

because u_2' is odd in η while g and q are even, we obtain

$$\frac{K_{22}}{k} = 1-\frac{2^{\frac{1}{2}}}{\pi}\phi\int_{-\eta_0}^{\eta_1}\int_0^{\infty}\left\{\frac{u_2'g}{q}\frac{\partial}{\partial\psi}\left[\int_0^{\eta}qg\,dz/\psi^{\frac{1}{2}}\right]-\frac{u_{20}'g_0}{q_0}\frac{\partial}{\partial\psi}\left[\int_0^{\eta}q_0g_0\,dz/\psi^{\frac{1}{2}}\right]\right\}d\psi\,d\eta, \quad (3.21)$$

where ϕ is the volume fraction of the particles.

Unfortunately, a careful examination of (3.21) reveals that it is not a convergent integral as η_0 and $\eta_1\to\infty$, its asymptotic form being

$$-\tfrac{1}{2}\{(\ln\eta_0)^2+(\ln\eta_1)^2\}+O(\ln\eta_0)+O(\ln\eta_1). \quad (3.22)$$

This can be shown by noting that, for $\psi\ll 1$, the integrand in (3.21) reduces, following integration by parts, to

$$-\psi^{-\frac{1}{2}}\left\{\frac{\partial}{\partial\psi}\left(\frac{u_2'g}{q}\right)\right\}\int_0^{\eta}qg\,d\eta. \quad (3.23)$$

However, when $|\eta|\gg 1$ and $\psi\ll 1$,

$$u_2' = -\frac{1}{2\eta^3},\quad q = \frac{1}{2^{\frac{1}{2}}|\eta|}(1+4\eta^2\psi)^{\frac{1}{2}},\quad g = 1$$

and

$$\int_0^{\eta}qg\,dz\to\frac{1}{2^{\frac{1}{2}}}\left(B+(1+4\eta^2\psi)^{\frac{1}{2}}+\ln\frac{|\eta|}{1+(1+4\eta\psi^2)^{\frac{1}{2}}}\right),$$

B being an $O(1)$ number. Therefore, when $|\eta|\gg 1$ and $\psi\ll 1$, (3.23) simplifies to

$$-\frac{2}{\psi^{\frac{1}{2}}(1+4\eta^2\psi)^{\frac{3}{2}}}\left(B+(1+4\eta^2\psi)^{\frac{1}{2}}+\ln\frac{|\eta|}{1+(1+4\eta^2\psi)^{\frac{1}{2}}}\right),$$

from which (3.22) follows readily.

The reason for the singularity in (3.21) can be traced to the simplification introduced in (3.7), where it was assumed that q could be replaced by q_0, which led to (3.19). As is evident, however, from the expression given above for q, this is only permissible provided that $|\eta| < O(Pe^{\frac{1}{6}})$. Thus, when $|\eta|\geqslant O(Pe^{\frac{1}{6}})$, the equation within the thermal layer becomes, in lieu of (3.19),

$$\frac{\partial \hat{T}}{\partial \bar{\eta}} = \frac{1}{2^{\frac{1}{2}}|\bar{\eta}|}\frac{\partial}{\partial \Psi}\left\{(1+4\bar{\eta}^2\Psi)^{\frac{1}{2}}\frac{\partial \hat{T}}{\partial \Psi}\right\}, \quad \hat{T} \equiv Pe^{\frac{1}{2}}T, \tag{3.24}$$

the solution to which, as $|\bar{\eta}| \to \infty$, can be shown to approach $(2\Psi)^{\frac{1}{2}}\mathscr{H}(\Psi)$ sufficiently rapidly for the integrals

$$\frac{1}{2^{\frac{1}{2}}}\int_{-\infty}^{-\bar{\eta}_0}\int_{-\infty}^{\infty}\frac{d\bar{\eta}\,d\Psi}{\bar{\eta}^2(1+4\bar{\eta}^2\Psi)^{\frac{1}{2}}}\{\hat{T}-(2\Psi)^{\frac{1}{2}}\mathscr{H}\}, \quad \frac{1}{2^{\frac{1}{2}}}\int_{\bar{\eta}_1}^{\infty}\int_{-\infty}^{\infty}\frac{d\bar{\eta}\,d\Psi}{\bar{\eta}^2(1+4\bar{\eta}^2\Psi)^{\frac{1}{2}}}\{\hat{T}-(2\Psi)^{\frac{1}{2}}\mathscr{H}\}, \tag{3.25}$$

with fixed $(\bar{\eta}_0, \bar{\eta}_1) \equiv Pe^{-\frac{1}{2}}(\eta_0, \eta_1)$, to exist. Furthermore, in view of (3.5),

$$T_1(\psi,\eta) \to T_1(\psi,0) + \frac{1}{4}\left\{\frac{B_1(\eta)}{\psi^{\frac{1}{2}}} - \frac{\ln\eta}{\psi^{\frac{1}{2}}}\right\} \quad \text{for} \quad \eta > 0,$$

$B_1(\eta)$ being $O(1)$; hence, because of the matching requirement between this outer solution and the temperature within the thermal layer, $\hat{T}$ in (3.24) and, therefore, the integrals (3.25) for fixed $\bar{\eta}_0$ and $\bar{\eta}_1$ are at most $O(\ln Pe)$. Since the expression for K_{22} must be independent of the choice of $\bar{\eta}_0$ and $\bar{\eta}_1$, it then follows that the leading term of (3.21) plus (3.25) equals $-[\ln Pe^{\frac{1}{2}}]^2$, i.e.

$$K_{22}/k = 1 + (2^{\frac{1}{2}}\phi)/16\pi)\{(\ln Pe)^2 + O(\ln Pe) + O(1)\}, \tag{3.26}$$

which completes our derivation of the asymptotic form of K_{22}.

To be sure, (3.26) is rather useless from the practical point of view since the $O[(\ln Pe)^2]$ term dominates the other two only if Pe is truly extremely large, but, unfortunately, the calculation of the remaining terms in (3.26) would require the solution of (3.24) subject to complicated boundary and matching conditions. This will not be attempted here and hence, again from a practical standpoint, (3.26) is incomplete. Nevertheless, it is felt that the analysis leading to (3.26) is of some interest, because it clearly brings out many of the physically significant features of this problem.

We complete our discussion with regard to a dilute suspension of cylinders in a simple shear by deriving the corresponding asymptotic expression, as $Pe \to \infty$, for K_{12}, which, in view of (3.11), requires the determination of

$$\int_{V_1-V^*} u_1' t'\, dV. \tag{3.27}$$

It can easily be shown from (3.1) and (3.4) though that the term

$$u_1'\{(2\psi)^{\frac{1}{2}} - x_2\} \to 2x_1^2x_2^2/r^6 + O(r^{-4})$$

as $r \to \infty$, and hence the integral (3.27) is *absolutely divergent* as the dimensions of V_1 are increased without bound. Hence it would seem rather unlikely that there exists a fluctuating quantity having the same asymptotic behaviour as $u_1' t$ but whose bulk value is known which could be used to convert (3.27) into an absolutely convergent integral as was done earlier with (3.15). This suggests in turn that, in the present case, particle–particle interactions cannot be neglected in deriving an expression for K_{12} even when $\phi \to 0$, and that the linear dimensions of V_1

must be taken as $O(\phi^{-\frac{1}{2}})$, rather than infinite as has been assumed up to now. Consequently, (3.27) becomes $-\frac{1}{8}\pi\ln\phi+O(1)$ and hence

$$K_{12}/k = \tfrac{1}{8}Pe\phi\{\ln\phi+O(1)\}. \tag{3.28}$$

It is of some interest that K_{12} is negative.

4. Spherical particles in a simple shear

The temperature field

The evaluation of the temperature field for the present problem is in most respects similar to that described in §3. Specifically, with $u_i = U_i = x_2\delta_{i1}$ at infinity, the streamlines are the intersection curves of the two families of surfaces of revolution (cf. Cox *et al.* 1968)

$$\left.\begin{aligned} E &= x_2^2/r^2f^2(r)-f_1(r), \quad -f_1(1)\leqslant E<\infty,\\ C &= x_3/rf(r), \quad -C_{\max}(E)\leqslant C\leqslant C_{\max}(E),\end{aligned}\right\} \tag{4.1}$$

where $$f(r) = [r^3-\tfrac{5}{2}+\tfrac{3}{2}r^{-2}]^{-\frac{1}{3}},\quad f_1(r) = \int_r^\infty \rho^{-3}f(\rho)\,d\rho.$$

Again there exist closed streamlines, which are formed when E lies in the interval $-f_1(1)\leqslant E<0$, where $E=-f_1(1)$ denotes the surface of the sphere and $E=0$ the so-called limiting stream surface. For $E\geqslant 0$ and $-\infty<C<\infty$ all streamlines are open and originate at upstream infinity.

As with the cylinder problem, at large Pe and with $\overline{T}=x_2$, the stream surfaces E are isothermal to a first approximation and hence, by analogy with the results of the previous section, we conclude that the temperature of the region of closed streamlines is zero except in a thin region near $E=0$. Thus (3.9) holds once again and, as in the two-dimensional case, the conductivity of the inclusion does not affect the bulk heat flux.

In the outer region, in which the stream surfaces originate at infinity, where $E=x_2^2$ and $\overline{T}=x_2$, we obtain, once again by analogy with (3.4) and (3.5),

$$T_0 = \pm E^{\frac{1}{2}} \tag{4.2}$$

and $$T_1 = T_1|_{\eta=0}+\int_0^\eta\{-\tfrac{1}{4}|\nabla E|^2E^{-\frac{3}{2}}+\tfrac{1}{2}(\nabla^2E)E^{-\frac{1}{2}}\}gh_E\,d\eta, \tag{4.3}$$

where the integration is along a streamline and g is the metric coefficient of the variable η. Of course, there exists also a thin region along the limiting stream surface where conduction normal to $E=0$ becomes comparable to convection along streamlines on that surface, and in which the dominant terms of the heat equation are

$$\frac{\partial T}{\partial\eta} = \frac{g}{qPe}\left\{|\nabla E|^2\frac{\partial^2T}{\partial E^2}+\nabla^2E\frac{\partial T}{\partial E}\right\}, \tag{4.4}$$

where ∇E, ∇^2E, q and g are functions of all three dimensions. It is evident that, except as noted later on, the first term on the right-hand side of (4.4) dominates the second within this thermal layer and thus (4.4) can be further reduced to

$$\frac{\partial T}{\partial\eta} = \frac{g_0}{q_0}|\nabla_0E|^2\frac{\partial^2T}{\partial\tilde{E}^2},\quad \tilde{E}=Pe^{\frac{1}{2}}E, \tag{4.5}$$

$$T\to Pe^{-\frac{1}{4}}\tilde{E}^{\frac{1}{2}}\quad\text{as}\quad\tilde{E}\to\infty,\qquad T\to 0\quad\text{as}\quad\tilde{E}\to-\infty.$$

The effective thermal conductivity

As in §3, we shall consider separately the evaluation of K_{12} and K_{22}. To begin with, we recall (3.9) and (3.10) and evaluate the integral within the spherical inclusion. Since, again, within the particle

$$t' = -x_2, \quad u_i' = -\tfrac{1}{2}(\delta_{i1}x_2 + \delta_{i2}x_1),$$

it is evident that

$$n\int_{V^*} u_i' t' \, dV^* = \frac{\phi}{10}\delta_{i1},$$

and therefore the expression for K_{i2} [analogous to (3.11)] becomes

$$\frac{K_{i2}}{k} = \delta_{i2} - \frac{\phi c_p^*}{10 c_p} Pe\, \delta_{i1} - nPe \int_{V_1 - V^*} u_i' t' \, dV. \tag{4.6}$$

We now proceed with the evaluation of K_{22}. The components of the disturbance velocity are given by (Cox *et al.* 1968)

$$u_i' = \frac{5(1-r^2)x_i x_2 x_1}{2r^7} - \frac{\delta_{i2}x_1 + \delta_{i1}x_2}{2r^5}, \quad r^2 = x_j x_j, \tag{4.7}$$

hence the contribution to the integral in (4.6) from the region of closed streamlines, where $-f_1(r) \leqslant E < 0$ and $t = -x_2$, is

$$-\int_V u_2' x_2 \, dV, \quad -f_1(1) \leqslant E < 0. \tag{4.8}$$

At large distances, $f(r) = r^{-1} + O(r^{-4})$ and $f_1(r) = (3r^3)^{-1} + O(r^{-6})$; hence, along the limiting stream surface $E = 0$,

$$x_2^2 = \frac{1}{3r^3} + O(r^{-6}), \quad u_2' = \frac{x_1}{2r^5}\{1 + O(r^{-3})\}.$$

It is now evident that the integral in (4.8) is absolutely convergent and its value is easily seen to be zero owing to the symmetries in the geometry of the region and the fact that u_2' is odd in x_1 and even in x_2 and x_3.

For the outer region we recall (4.2) and (4.3), which, in analogy with (3.14), give for the upper half-space

$$t' = E^{\frac{1}{2}} - x_2 + \frac{1}{Pe}\left\{T_1|_{\eta=0} + \int_0^\eta [-\tfrac{1}{4}|\nabla E|^2 E^{-\frac{3}{2}} + \tfrac{1}{2}(\nabla^2 E)E^{-\frac{1}{2}}] g h_E \, d\eta\right\}. \tag{4.9}$$

The first-order contribution to K_{22} arises from

$$\int_V u_2'(E^{\frac{1}{2}} - x_2)\, dV, \quad E \geqslant 0, \tag{4.10}$$

in which, because of (4.1) and (4.7), the integrand is $O(r^{-6})$ for $E > 0$ and $O(r^{-\frac{11}{2}})$ along $E = 0$ as $r \to \infty$. Evidently (4.10) is absolutely convergent and has value zero.

It appears then that, as in the two-dimensional problem, the leading contribution to K_{22} results from

$$\int_V u_2' T_1 \, dV, \quad E \geqslant 0, \tag{4.11}$$

plus the appropriate integral from the thermal layer that exists along the surface $E = 0$. Although the algebra is here somewhat different, the steps required to evaluate (4.11) using (4.3) and (4.5) are similar to those employed earlier in deriving (3.21) from (3.14) and (3.19). Thus we obtain

$$\frac{K_{22}}{k} = 1 - \frac{3\phi}{4\pi} \int_{-\eta_0}^{\eta_1} \int_{-\eta_0}^{\eta_1} \int_0^{\infty} \left[\frac{h_c h_E g u_2'}{E^{\frac{3}{2}}} \int_0^{\eta} \{ -\tfrac{1}{4} |\nabla E|^2 + \tfrac{1}{2} |\nabla^2 E| \, E \} g h_E \, dz \right.$$
$$\left. - \frac{h_{c0} h_{E0} g_0 u_{20}'}{E^{\frac{3}{2}}} \int_0^{\eta} - \tfrac{1}{4} |\nabla_0 E| \, \frac{2 g_0 h_{E0}}{q_0} \, dz \right] dE \, d\eta \, dC, \tag{4.12}$$

where, again, the zero subscripts indicate quantities evaluated at $E = 0$ and h_c is the metric coefficient of the co-ordinate C. There will be no contribution from the integral

$$\int_V u_2' T_1 |_{\eta=0} \, dV, \quad E > 0, \tag{4.13}$$

which converges absolutely and whose value is zero because of symmetry.

Now, a careful examination of (4.12) reveals that, again, this is not a convergent integral when η_0 and $\eta_1 \to \infty$, its form being given by

$$-A\{\eta_1^{\frac{1}{2}} + \eta_0^{\frac{1}{2}}\} + O(1), \tag{4.14}$$

where A is a positive $O(1)$ number. The above can be obtained by employing the asymptotic forms, for $|\eta| \gg 1$ and $E \ll 1$,

$$\left. \begin{aligned} u_2' &= -\frac{[1 - C^2/\eta^2]^{\frac{1}{2}}}{2\eta^4}, \quad q = \frac{1}{3^{\frac{1}{2}} |\eta|^{\frac{3}{2}}} (1 + 3E|\eta|^3)^{\frac{1}{2}}, \\ h_E &= \frac{3^{\frac{1}{2}}}{2} \frac{|\eta|^{\frac{3}{2}}}{(1 + 3E|\eta|^3)^{\frac{1}{2}}}, \quad |\nabla E| = \frac{2}{3^{\frac{1}{2}} \eta^{\frac{3}{2}}} (1 + 3E|\eta|^3)^{\frac{1}{2}}, \quad \nabla^2 E = 2, \\ h_c &= (1 - C^2/\eta^2)^{-\frac{1}{2}}, \quad g = (1 - C^2/\eta^2)^{-\frac{1}{2}}, \end{aligned} \right\} \tag{4.15}$$

in terms of which the integrand in (4.12) that gives the leading contribution to the $O(\phi)$ term in K_{22} simplifies to (for $\eta > 0$)

$$- \frac{1}{4E^{\frac{3}{2}} |\eta|^{\frac{5}{2}} (1 + 3E|\eta|^3)^{\frac{1}{2}} (1 - C^2/\eta^2)^{\frac{1}{2}}} \int_C^{\eta} \frac{dz}{z^{\frac{3}{2}} (1 - C^2/z^2)^{\frac{1}{2}} (1 + 3Ez^3)^{\frac{1}{2}}}$$
$$+ \frac{1}{4E^{\frac{3}{2}} |\eta|^{\frac{5}{2}} (1 - C^2/\eta^2)^{\frac{1}{2}}} \int_C^{\eta} \frac{dz}{z^{\frac{3}{2}} (1 - C^2/z^2)^{\frac{1}{2}}}.$$

Equation (4.14) then follows directly; the coefficient A must of course be evaluated numerically.

It is again evident that the approximations introduced in reducing (4.4) to (4.5) are not valid everywhere in the thermal layer. In fact, (4.5) applies only as long as $3E|\eta|^3 \ll 1$. When this is no longer the case, i.e. when $|\eta| \gg 1$ and $E|\eta|^3$

is $O(1)$, we obtain, in lieu of (4.4) and in view of (4.15),

$$\frac{\partial T}{\partial \eta} = \frac{4(1-C^2/\eta^2)^{\frac{1}{2}}}{3^{\frac{1}{2}}|\eta|^{\frac{3}{2}}Pe}\frac{\partial}{\partial E}\left\{(1+3E|\eta|^3)^{\frac{1}{2}}\frac{\partial T}{\partial E}\right\}. \tag{4.16}$$

Clearly, when $E|\eta|^3$ is $O(1)$, the thickness of the thermal layer can no longer be $O(Pe^{-\frac{1}{3}})$ since, in that case, conduction would play a secondary role in (4.16). Therefore, in order to maintain a proper balance between the conductive and the convective term of the energy equation, it is necessary to transform (4.16) into

$$\frac{\partial \hat{T}}{\partial \hat{\eta}} = \frac{4(1-\hat{C}^2\hat{\eta}^{-2})^{\frac{1}{2}}}{3^{\frac{1}{2}}|\hat{\eta}|^{\frac{3}{2}}}\frac{\partial}{\partial \hat{E}}\left\{(1+3\hat{E}|\hat{\eta}|^{\frac{3}{2}})^{\frac{1}{2}}\frac{\partial \hat{T}}{\partial \hat{E}}\right\}, \tag{4.17}$$

where $\hat{E} = Pe^{\frac{6}{11}}E$, $\hat{T} \equiv Pe^{\frac{3}{11}}T$ and $(\hat{\eta}, \hat{C}) = Pe^{-\frac{2}{11}}(\eta, C)$. By analogy with (3.24), the corresponding equation of the cylinder problem of §3, it appears safe to assume that, again, as $|\hat{\eta}| \to \infty$ the solution to (4.17) approaches $\hat{E}^{\frac{1}{2}}\mathscr{H}(\hat{E})$ sufficiently rapidly for the integral in the expression for K_{22} containing the far-field temperature distribution along the limiting stream surface $E = 0$ to exist. This being the case, it is then an easy matter to estimate the order of magnitude of this integral using the transformations given by (4.17) and thereby show that

$$K_{22}/k = 1 + \phi\{A_1 Pe^{\frac{1}{11}} O(1)\}, \tag{4.18}$$

where A_1 is a numerical coefficient to be evaluated from the solution of (4.17).

In contrast to the difficulties encountered in the two-dimensional case, the integral

$$\int_{V_1 - V^*} u_1'(T_0 - x_2)\, dV_1$$

is here absolutely convergent. K_{12} then becomes

$$\frac{K_{12}}{k} = -\frac{3\phi Pe}{4\pi}\int_{V_1 - V^*} u_1'(T_0 - x_2)\, dV_1 - \frac{\phi c_p^*}{10 c_p} Pe, \tag{4.19}$$

or

$$K_{12}/k = -\phi Pe\{0{\cdot}578 + c_p^*/10c_p).$$

Again, as with (3.28), this non-diagonal component of K_{ij} is negative.

The analysis presented in §§3 and 4 applies of course only to cases in which the particle Reynolds number Re is sufficiently small for inertia effects to be negligible. Naturally, the presence of such effects would have a profound influence on the structure of the flow field (cf. Robertson & Acrivos 1970; Lin, Peery & Schowalter 1970) and would introduce additional terms in the expression for K_{22}, e.g. terms $O(Pe\,Re \ln Re)$ and $O(Pe\,Re)$ in the two-dimensional problem and terms $O(Pe\,Re)$ and $O(Pe\,Re^{\frac{3}{2}})$ in the three-dimensional case, and perhaps even more important terms. Another point to consider would be the time required to reach a steady state following a temperature disturbance in the free stream, because at high Pe the appropriate time constant within the region of closed streamlines should be quite substantial. The influence of these and other effects on the results of the present analysis remain, however, to be evaluated.

This work was supported in part by grants from the National Science Foundation, GK-36515X and GK-41781. The authors are grateful to Dr E. J. Hinch of Cambridge University for his many helpful comments on earlier versions of this paper.

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